Hybrid & Alternative Logics in Isabelle:

Isabelle/Set

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Isabelle/Set

A softly-typed higher-order set-theoretic logic for Isabelle

Ongoing work with

Alexander Krauss
Cezary Kaliszyk
Karol Pąk

In this talk

Set theory for formal proof

History & new ideas

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Soft types

Extended typing functionality for HOL

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Overview, aims, & current work

Set Theory for Formal Proof

History

Used from the beginning: Metamath, Mizar.

In Isabelle: ZF, HOLZF.

Large math libraries formalized.

More recently

Calls for a renaissance of set theory in formal proof.

Isabelle/Mizar [Kaliszyk, Pąk '18]: Mizar semantics in Isabelle. First ~100 MML articles verified.

auto2 [Zhan '17]: formalization of the fundamental group in untyped ZFC from scratch.

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- FOL standard (Mizar, Metamath, Isabelle/ZF)
- But axiom schemas!

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Isabelle/HOL snippet:

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empty_axiom: "¬(∃x. x ∈ {})" and
elem_induct_axiom: "(∀X. (∀x. x ∈ X → P x) → P X) → (∀X. P X)" and
Union_axiom: "∀X x. x ∈ ⋃X ↔ (∃Y. Y ∈ X ∧ x ∈ Y)" and
Replacement_axiom: "∀X y. y ∈ Repl X F ↔ (∃x. x ∈ X ∧ y = F x)" and
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Has a model under reasonable assumptions [Brown, Pąk, Kaliszyk '19].

Also the foundation of Chad Brown's Egal theorem prover.

Soft Types

Types & predicates

In type theory, types:

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In untyped formalisms, predicates do this.

Predicates & types

Example: Predicates

```
∀G x y z.
  (is_monoid G) → (x ∈ carrier G) → (y ∈ carrier G) → (z ∈ carrier G) →
    (x ∗ y = x ∗ z) → (x ∈ units G) → y = z
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Abstracting such predicates as "soft types" improves structure and automation.

Example: Soft types

```
∀G: monoid. ∀x y z: element G. (x * y = x * z) → (x ∈ units G) → y = z
```

Implementation

A generic layer on top of Isabelle/HOL.

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Type constructions:

Base types	`Type P`
Dependent functions	`(x: A) ⇒ B x`
Type intersections	`A ¦ B`
Type adjectives	`adj · type`

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Working on type elaboration, type derivation, and integrating type reasoning.

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Strong and structured soft type system and automation.

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Aims

Provide a simpler, modern base to import the MML.

Maintain Isabelle/ZF compatibility.

Formalize more stuff!

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Current work

Type derivation

Structures

Set extensions

Fin.

Thanks for listening!

References I

- Chad E. Brown, Cezary Kaliszyk, and Karol Pąk. *Higher-order Tarski Grothendieck as a Foundation for Formal Proof*. Accepted for ITP 2019.
- Cezary Kaliszyk and Karol Pąk. "Semantics of Mizar as an Isabelle Object Logic". In: *Journal of Automated Reasoning* (Aug. 2018). ISSN: 1573-0670. DOI: 10.1007/s10817-018-9479-z. URL: https://doi.org/10.1007/s10817-018-9479-z.
- Alexander Krauss. *Adding Soft Types to Isabelle*. https://www21.in.tum.de/~krauss/papers/soft-types-notes.pdf. Unpublished note. 2010.
- Leslie Lamport and Lawrence C. Paulson. "Should Your Specification Language Be Typed?" In: *ACM Trans. Program. Lang. Syst.* 21.3 (May 1999), pp. 502–526. ISSN: 0164-0925. DOI: 10.1145/319301.319317. URL: http://doi.acm.org/10.1145/319301.319317.

References II

Bohua Zhan. "Formalization of the fundamental group in untyped set theory using auto2". In: *CoRR* abs/1707.04757 (2017). arXiv: 1707.04757. URL: http://arxiv.org/abs/1707.04757.