Semisimplicial Types in Internal Categories with Families

Joshua Chen and Nicolai Kraus

University of Nottingham, United Kingdom

Abstract

An open question in homotopy type theory, known as the problem of *constructing semisimplicial types*, is whether one can define a function $\mathsf{SST} : \mathbb{N} \to \mathsf{Type}_1$ such that $\mathsf{SST}(n)$ is the type of all configurations of triangles and tetrahedra of dimension up to n. We show in Agda that semisimplicial types can be constructed in any set-based internal category with families that contains Σ, Π, and a universe. This means that, given a CwF $(\mathsf{Con}, \mathsf{Ty}, \ldots)$, we construct a function $\mathsf{SST}_\mathsf{c} : \mathbb{N} \to \mathsf{Con}$. This project is work in progress.

Semisimplicial types. *Constructing semisimplicial types* [Uni13, Her15] is an open problem in homotopy type theory and, more generally, in dependent type theory without UIP. It was first discussed between Voevodsky, Lumsdaine and others during the Univalent Foundations special year at the IAS Princeton in 2012–13. In the following, we describe the problem.

A semisimpicial type of dimension 2 is a tuple (A_0, A_1, A_2), where $A_0 : \mathsf{Type}$ is a type of points, $A_1 : A_0 \to A_0 \to \mathsf{Type}$ is a type of lines (for any two points), and $A_2 : (x\,y\,z : A_0) \to A_1\,x\,y \to A_1\,y\,z \to A_1\,x\,z \to \mathsf{Type}$ is a type of triangle fillers (for any three points and three lines forming a triangle). Similarly, a *semisimplicial type of dimension* n should be a tuple $(A_0, \ldots, A_n)$ which represents collections of triangles and tetrahedra of dimensions at most n. The open problem asks: can one can define a function $\mathsf{SST} : \mathbb{N} \to \mathsf{Type}_1$ such that $\mathsf{SST}(n)$ is equivalent to the (record/Σ-) type of such tuples $(A_0, \ldots, A_n)$?

Construction in internal CwF's. By an *internal CwF*, we mean a type $\mathsf{Con} : \mathsf{Type}$ together with families $\mathsf{Sub} : \mathsf{Con} \to \mathsf{Con} \to \mathsf{Type}$, $\mathsf{Ty} : \mathsf{Con} \to \mathsf{Type}$, $\mathsf{Tm} : (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\,\Gamma \to \mathsf{Type}$, and all the components and equalities which are needed to define a category with families [Dyb95]. We say that such a CwF is *set-based* if Con, Ty, Sub, Tm are families of sets in the sense of homotopy type theory, i.e. types satisfying the principle UIP.

The goal of this project is to define, for any set-based internal CwF with Π and Σ-types and a universe U, a function $\mathsf{SST}_\mathsf{c} : \mathbb{N} \to \mathsf{Con}$ in Agda such that $\mathsf{SST}_\mathsf{c}\ n$ is the context $(A_0 : U, A_1 : A_0 \to A_0 \to U, \ldots, A_n : \ldots)$.

A connection between open problems. A second open question, originally asked by Shulman [Shu14], is whether type theory without UIP can model ("eat" [Cha09]) itself. More concretely, the problem is to formalise the syntax of type theory inside type theory as a set-based CwF and to give interpretation functions which send the syntax to actual types and their elements in the "obvious" way: for example, a context in the CwF should be interpreted as the nested Σ-type of all its components. For a detailed discussion, see the introduction of [Kra21].

Our current project shows how a solution of this question would give rise to a solution to the open problem of constructing semisimplicial types, as claimed by Shulman [Shu14]: composing SST_c with the interpretation $\mathsf{Con} \to \mathsf{Type}_1$, we would get the desired function $\mathbb{N} \to \mathsf{Type}_1$.

Related work. We are aware of two different scripts which, when given a number n as input, produce valid Agda code for $\mathsf{SST}(n)$—one script using Haskell [Kra14] and one using Python [Bru]. Our function SST_c can be seen as a dependently typed version of such a script,

with Agda replaced by an internal CwF and Haskell/Python replaced by Agda. Since Haskell and Python simply produce strings while SST_c is required to type-check, the latter is significantly more difficult to define and requires several new ideas.

Our work can also be seen as a variation of the construction of semisimplicial types in Voevodsky's *homotopy type system* [Voe13] or *2LTT (2-level type theory)* [ACK16, ACKS19]. Here, Agda plays the role of the outer ("strict") theory and the internal CwF the role of the inner ("fibrant") one. However, our internal CwF is too minimalistic (e.g. no finite types) to mimic the direct construction of [ACK16]. Moreover, 2LTT makes it possible to first formulate a type in the strict theory and prove its fibrancy afterwards, a strategy which is not possible in our setting.

It is known that semisimplicial *sets* can be constructed in homotopy type theory, i.e. we can define a function $\mathsf{SST}_0 \colon \mathbb{N} \to \mathsf{Type}_1$ which only considers *sets* of points, *sets* of lines, and so on. Although our CwF's are based on sets, this is unrelated; our CwF's are not assumed to have an identity type, and therefore the notion of truncatedness does not exist for internal types.

Formalisation of the construction. Using the HoTT-Agda library [SH12] we formalise set-based CwF's as records with fields Con, Sub, Ty, Tm, the usual operations thereon, and equations given by their usual presentation as a generalized algebraic theory (see e.g. Fig. 1 of [Kra21]). Since we are motivated by the goal of internalizing constructions in generic homotopy type theory, where many CwF's (such as the formalised syntax proposed by Altenkirch and Kaposi [AK16]) do not satisfy additional definitional equalities, we avoid any use of rewriting pragmas. This results in a proliferation of transports, which we mitigate to a small extent using instance resolution.

Our CwF's are further equipped with internal type formers $\hat{\Pi}$ and $\hat{\Sigma}$, as well as a family U of base types (polymorphic over contexts Γ) together with decoding function $\mathsf{el} \colon \mathsf{Tm}\, U \to \mathsf{Ty}\, \Gamma$. We then define $\mathsf{SST}_\mathsf{c}\, 0 \coloneqq U$ and $\mathsf{SST}_\mathsf{c}\,(n+1) \coloneqq \big(\mathsf{SST}_\mathsf{c}\, n, (\mathsf{sk}\,(n+1) \mathrel{\hat{\to}} U)\big)$ by mutual induction with $\mathsf{sk} \colon (n \colon \mathbb{N}) \to \mathsf{Ty}\,(\mathsf{SST}_\mathsf{c}\,(n-1))$, where $\mathrel{\hat{\to}}$ is the function type in the internal CwF. The main difficulty lies in defining the type $\mathsf{sk}\, n$ of $\partial\Delta^n$-shaped tuples indexed over $\mathsf{SST}_\mathsf{c}\,(n-1)$, for $n \geq 1$.

A high-level description of our approach to this is as follows. We define $\mathsf{sk}\,(n+1) \coloneqq \mathsf{shape}\big(n+1, n, \binom{n+2}{n+1}\big)$, where

$$\mathsf{shape} \colon (b\, h\, t \colon \mathbb{N}) \to \mathsf{Ty}\,(\mathsf{SST}_\mathsf{c}\, h) \quad \text{for } 0 \leq h < b,\ 1 \leq t \leq \tbinom{b+1}{h+1}$$

encodes, as an internal nested $\hat{\Sigma}$-type $\mathsf{shape}\, b\, h\, t$, the subfunctor of the representable functor $\Delta_+[b]$[1] which omits all arrows $[i] \to [b]$ for $i > h$, as well as those arrows $[h] \to [b]$ above the t-th (ordered via a bijection $\varphi \colon \mathsf{Fin}\tbinom{b+1}{h+1} \cong \Delta_+([h],[b])$). More concretely, its elements may be seen as tuples corresponding to certain simplicial subcomplexes of the b-simplex filled by $\mathsf{SST}_\mathsf{c}\, h$. This allows us to define, by induction on h and t,

$$\mathsf{shape}\, b\, h\, t \coloneqq \hat{\Sigma}[\sigma \colon \mathsf{shape}\, b\, h'\, t']\,\big(A_h(\mathsf{inter}\,\sigma\,\varphi(t))\big),$$

where (h', t') is the obvious predecessor of (h, t) and $\mathsf{inter}\,\sigma\, f$ picks out the subtuple of σ corresponding to the arrow f. It remains to construct this intersection function inter, again by induction on the indices b, h and the dimension k of f, and at this point we will need to use that our CwF is set-truncated.

Source code. Our work in progress is available at `github.com/jaycech3n/CwF`.

[1] $\Delta_+ \coloneqq \Delta$ without degeneracies.

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